## Calculating Determinants

- The determinant of a $2 \times 2$ matrix

$$
M=\left(\begin{array}{ll}
m_{11} & m_{12}  \tag{1}\\
m_{21} & m_{22}
\end{array}\right)
$$

is given by

$$
|M|=\left|\begin{array}{ll}
m_{11} & m_{12}  \tag{2}\\
m_{21} & m_{22}
\end{array}\right|=m_{11} m_{22}-m_{12} m_{21}
$$

- The determinant of a $3 \times 3$ matrix

$$
M=\left(\begin{array}{lll}
m_{11} & m_{12} & m_{13}  \tag{3}\\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{array}\right)
$$

can be written in terms of the determinants of $2 \times 2$ sub-matrices

$$
|M|=\left|\begin{array}{lll}
m_{11} & m_{12} & m_{13}  \tag{4}\\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{array}\right|=m_{11}\left|\begin{array}{cc}
m_{22} & m_{23} \\
m_{32} & m_{33}
\end{array}\right|-m_{12}\left|\begin{array}{cc}
m_{21} & m_{23} \\
m_{31} & m_{33}
\end{array}\right|+m_{13}\left|\begin{array}{cc}
m_{21} & m_{22} \\
m_{31} & m_{32}
\end{array}\right|
$$

- In general, each element of the top row of the matrix is multiplied by the determinant of the sub-matrix obtained by removing the row and column containing that element. The results are then added together with alternating sign, starting with a positive $m_{11}$ term. Figure 1 shows the top-row elements and the associated sub-matrices and signs for the $2 \times 2,3 \times 3$, and $4 \times 4$ cases.

| $(+)$ | $(-)$ | $(+)$ | $(-)$ |
| :--- | :--- | :--- | :--- | :--- |
| $\mathrm{m}_{11}$ | $\mathrm{~m}_{12}$ | $\mathrm{~m}_{11}$ | $\mathrm{~m}_{12}$ |
| $\mathrm{~m}_{21}$ | $\mathrm{~m}_{22}$ | $\mathrm{~m}_{21}$ | $\mathrm{~m}_{22}$ |


| $(+)$ | $(-)$ | $(+)$ | $(+)$ | $(-)$ | $(+)$ | $(+)$ | $(-)$ | $(+)$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{m}_{11}$ | $\mathrm{~m}_{12}$ | $\mathrm{~m}_{13}$ | $\mathrm{~m}_{11}$ | $\mathrm{~m}_{12}$ | $\mathrm{~m}_{13}$ | $\mathrm{~m}_{11}$ | $\mathrm{~m}_{12}$ | $\mathrm{~m}_{13}$ |
| $\mathrm{~m}_{21}$ | $\mathrm{~m}_{22}$ | $\mathrm{~m}_{23}$ | $\mathrm{~m}_{21}$ | $\mathrm{~m}_{22}$ | $\mathrm{~m}_{23}$ | $\mathrm{~m}_{21}$ | $\mathrm{~m}_{22}$ | $\mathrm{~m}_{23}$ |
| $\mathrm{~m}_{31}$ | $\mathrm{~m}_{32}$ | $\mathrm{~m}_{33}$ | $\mathrm{~m}_{31}$ | $\mathrm{~m}_{32}$ | $\mathrm{~m}_{33}$ | $\mathrm{~m}_{31}$ | $\mathrm{~m}_{32}$ | $\mathrm{~m}_{33}$ |


| $(+)$ | $(-)$ | $(+)$ | $(-)$ | $(+)$ | $(-)$ | $(+)$ | $(-)$ | $(+)$ | $(-)$ | $(+)$ | $(-)$ | $(+)$ | $(-)$ | $(+)$ | $(-)$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{m}_{11}$ | $\mathrm{~m}_{12}$ | $\mathrm{~m}_{13}$ | $\mathrm{~m}_{14}$ | $\mathrm{~m}_{11}$ | $\mathrm{~m}_{12}$ | $\mathrm{~m}_{13}$ | $\mathrm{~m}_{14}$ | $\mathrm{~m}_{11}$ | $\mathrm{~m}_{12}$ | $\mathrm{~m}_{13}$ | $\mathrm{~m}_{14}$ | $\mathrm{~m}_{11}$ | $\mathrm{~m}_{12}$ | $\mathrm{~m}_{13}$ | $\mathrm{~m}_{14}$ |
| $\mathrm{~m}_{21}$ | $\mathrm{~m}_{22}$ | $\mathrm{~m}_{23}$ | $\mathrm{~m}_{24}$ | $\mathrm{~m}_{21}$ | $\mathrm{~m}_{22}$ | $\mathrm{~m}_{23}$ | $\mathrm{~m}_{24}$ | $\mathrm{~m}_{21}$ | $\mathrm{~m}_{22}$ | $\mathrm{~m}_{23}$ | $\mathrm{~m}_{24}$ | $\mathrm{~m}_{21}$ | $\mathrm{~m}_{22}$ | $\mathrm{~m}_{23}$ | $\mathrm{~m}_{24}$ |
| $\mathrm{~m}_{31}$ | $\mathrm{~m}_{32}$ | $\mathrm{~m}_{33}$ | $\mathrm{~m}_{34}$ | $\mathrm{~m}_{31}$ | $\mathrm{~m}_{32}$ | $\mathrm{~m}_{33}$ | $\mathrm{~m}_{34}$ | $\mathrm{~m}_{31}$ | $\mathrm{~m}_{32}$ | $\mathrm{~m}_{33}$ | $\mathrm{~m}_{34}$ | $\mathrm{~m}_{31}$ | $\mathrm{~m}_{32}$ | $\mathrm{~m}_{33}$ | $\mathrm{~m}_{34}$ |
| $\mathrm{~m}_{41}$ | $\mathrm{~m}_{42}$ | $\mathrm{~m}_{43}$ | $\mathrm{~m}_{44}$ | $\mathrm{~m}_{41}$ | $\mathrm{~m}_{42}$ | $\mathrm{~m}_{43}$ | $\mathrm{~m}_{44}$ | $\mathrm{~m}_{41}$ | $\mathrm{~m}_{42}$ | $\mathrm{~m}_{43}$ | $\mathrm{~m}_{44}$ | $\mathrm{~m}_{41}$ | $\mathrm{~m}_{42}$ | $\mathrm{~m}_{43}$ | $\mathrm{~m}_{44}$ |

Figure 1: A visual guide to computing the determinants of $2 \times 2,3 \times 3$, and $4 \times 4$ matrices.

## An n $=2$ Example

$$
\begin{gather*}
M=\left(\begin{array}{cc}
3 & -1 \\
5 & 7
\end{array}\right)  \tag{5}\\
|M|=3 \cdot 7-(-1) \cdot 5=21+5=26 \tag{6}
\end{gather*}
$$

## An n $=3$ Example

$$
\begin{gather*}
M=\left(\begin{array}{ccc}
4 & 2 & 5 \\
-1 & 6 & 7 \\
3 & 1 & 2
\end{array}\right)  \tag{7}\\
|M|=4\left|\begin{array}{cc}
6 & 7 \\
1 & 2
\end{array}\right|-2\left|\begin{array}{cc}
-1 & 7 \\
3 & 2
\end{array}\right|+5\left|\begin{array}{cc}
-1 & 6 \\
3 & 1
\end{array}\right|  \tag{8}\\
=4(6 \cdot 2-7 \cdot 1)-2((-1) \cdot 2-7 \cdot 3)+5((-1) \cdot 1-6 \cdot 3)=20+46-95=-29 \tag{9}
\end{gather*}
$$

This work is licensed under the Creative Commons Attribution-ShareAlike 4.0 International License: http: //creativecommons.org/licenses/by-sa/4.0/.
L.A. Riley (lriley@ursinus.edu), updated June 2021

